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1974 J. Phys. A: Math. Nucl. Gen. 7 889

(http://iopscience.iop.org/0301-0015/7/8/003)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 8, 1974. Printed in Great Britain. © 1974

Nonlinear chiral dispersive waves

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Received 20 December 1973

Abstract. Whitham's theory of nonlinear water waves is applied to a classical field with the lagrangian density $\mathscr{L} = \frac{1}{2}\{[(\partial^\mu\phi)(\partial_\mu\phi) - m^2\phi^2]/(1+\lambda\phi^2)\}$. This is the isoscalar analogue of a chiral invariant SU(2) ⊗ SU(2) lagrangian with symmetry breaking term included. The corresponding field equation admits simple harmonic plane-wave solutions. We find that the important field quantities of these waves, namely the wavenumber k and amplitude A obey a system of first-order partial differential equations. When the coupling parameter λ is negative in sign, the system is hyperbolic, which implies that any inhomogeneities in k and A propagate with certain (amplitude-dependent) velocities. These velocities, which are the nonlinear generalization of the group velocity in the Whitham sense, are calculated.

1. Introduction

Considerable difficulties are encountered in the evaluation of the S matrix elements in quantum field models involving chirally invariant lagrangians or other non-polynomial lagrangians with and without derivative couplings, due to the occurrence of infinite number of diagrams in each order of perturbation. Various covariant summability techniques have been prescribed by many authors recently (for example, Faddeev and Slavanov 1973 and references therein). On the other hand, consideration of the classical field equations of motion also leads to useful observations. For instance, the application of the method of characteristics (Velo and Zwanziger 1969, Mathews and Seetharaman 1973) to various field equations resulted in a number of interesting observations regarding the causality property of propagation. It is the aim of the present paper (and papers to follow) to study the wave propagation properties of certain typical nonlinear field models of current interest in quantum field theory, by taking advantage of the recent interesting developments in the nonlinear wave problems of fluid and plasma dynamics.

There exists a recently well developed theory of nonlinear water waves by Whitham (1965, 1967), Lighthill (1965, 1967) and others (Hayes 1973 and references therein). This theory is being successfully applied to various problems in fluid and plasma dynamics. We adopt in this paper essentially Whitham's original method (Whitham 1965), where for the first time the concept of group velocity has been extended to nonlinear wave problems.

Whitham's theory is applicable to any field problem, if one has periodic steady-profile solutions. Of course not all the field problems of interest in quantum field theory have simple elementary wave solutions. But at least in certain typical cases we do have elementary plane-wave solutions which may be handled easily. The $\lambda\phi^4$ field contains such solutions (Mathews and Lakshmanan 1973). Another model with the lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\frac{(\partial^\mu\phi)(\partial_\mu\phi)-m^2\phi^2}{1+\lambda\phi^2}\right) \tag{1}$$

which we recently reported (Mathews and Lakshmanan 1974a) has very simple wave solutions. In the massless case this is the isoscalar analogue of the chiral invariant $SU(2)\times SU(2)$ lagrangian in the Gasiorowicz–Geffen coordinates (Delbourgo *et al* 1969)

$$\mathscr{L} = \frac{1}{2}\left((\partial^\mu\boldsymbol{\phi}).(\partial_\mu\boldsymbol{\phi})-\lambda\frac{(\boldsymbol{\phi}.\partial^\mu\boldsymbol{\phi})(\boldsymbol{\phi}.\partial_\mu\boldsymbol{\phi})}{1+\lambda\boldsymbol{\phi}^2}\right). \tag{2}$$

In this paper we study the propagation and group velocity properties of this model (1) using Whitham's technique (without taking recourse to Whitham's W function). In a subsequent paper the results of the ϕ^4 field model will be reported. In § 2 we give a brief account of Whitham's theory relevant for our purpose. In § 3 we establish the system of quasi-linear partial differential equations for the wavenumber k and amplitude A starting from a set of two conservation equations. We also show how the conservation of waves comes about in a natural way. In § 4 by an application of the method of characteristics we investigate the group velocity property of the waves and show that only when $\lambda < 0$ the above system of partial differential equations is hyperbolic. In this case the changes in k and A propagate with two different velocities whose riemannian invariant forms are also found. Both of these group velocities have the correct $\lambda \to 0$ limit.

2. Whitham's theory

Whitham's theory (Whitham 1965) is essentially based on the observation that even though exact general solutions of nonlinear wave equations are out of the question for the present, plane-wave solutions may always be given in the form

$$\phi = \Phi(X;\omega,k,A_i) \tag{3}$$

where

$$X = kx-\omega t, \qquad \omega = \omega(k,A_i). \tag{4}$$

(For simplicity we restrict ourselves to the one space–one time dimensional case.) He shows that for a more general category of solutions consisting of those which can be approximated by plane waves *locally* (so that they can be represented by (3) and (4) with k, A_i replaced by the slowly varying functions $k(x,t)$, $A_i(x,t)$ which vary little over several wavelengths) the temporal behaviour, eg, the motion of a 'wave packet', can be deduced from a consideration of conservation equations in the following way.

From any governing equation of motion of a system, one may write down a number of exact conservation equations of the form

$$\frac{\partial P}{\partial t}+\frac{\partial S}{\partial x} = 0. \tag{5}$$

Then these conservation equations may be averaged as

$$\frac{\partial \tilde{P}}{\partial t}+\frac{\partial \tilde{S}}{\partial x} = 0 \tag{6}$$

where the quantity

$$\tilde{F}(x,t) = \frac{1}{2\xi}\int_{x-\xi}^{x+\xi} F(x',t)\,\mathrm{d}x'. \tag{7}$$

If the wave train is approximately uniform in the distance 2ξ, the mean quantities $\tilde{P}$ and $\tilde{S}$ may be calculated from the uniform solution (3) and (4), holding k and A_i constant. Assuming that the interval $x-\xi < x' < x+\xi$ includes a small number of waves, these mean values will be functions of k and A_i alone. Thus the exact equation (5) is replaced by the approximate equation

$$\frac{\partial}{\partial t}\bar{P}(k, A_i) + \frac{\partial}{\partial x}\bar{S}(k, A_i) = 0, \tag{8}$$

where $\bar{P}$, $\bar{S}$ are calculated to be the mean value over a single wavelength in terms of the steady-profile solution (3) with (4):

$$\bar{P}(k, A_i) \equiv \frac{1}{\lambda}\int_0^{\lambda} \mathscr{P}\{\Phi(X; k, A_i)\}\,\mathrm{d}X, \tag{9}$$

where $\mathscr{P}\{\Phi\}$ denotes the function P in terms of (3).

Choosing the appropriate number of conservation equations one obtains a system of partial differential equations for the dependent variables k and A_i. In particular for the problems considered by Whitham and others the equations are hyperbolic and homogeneous in the derivatives even when the equations for the field ϕ are not. Then the propagation of these important physical quantities is described by the theory of characteristics. The generalized group velocities for the nonlinear problems are defined as the characteristic velocities of the above system of hyperbolic differential equations and these are the propagation velocities for the changes in a wave train.

3. The model

The Euler–Lagrange equation of motion corresponding to the lagrangian (1) is

$$(1+\lambda\phi^2)\partial^\mu\partial_\mu\phi + m^2\phi - \lambda\phi(\partial^\mu\phi)(\partial_\mu\phi) = 0. \tag{10}$$

The elementary Lorentz-invariant steady-profile solution of the form

$$\phi = \phi(\omega t - \boldsymbol{k}\,.\,\boldsymbol{x}) \tag{11}$$

is known (Mathews and Lakshmanan 1974a) to be (apart from an unimportant initial phase)

$$\phi = A\sin(\omega t - \boldsymbol{k}\,.\,\boldsymbol{x}) \tag{12}$$

with the amplitude-dependent dispersion relation

$$\omega^2 - k^2 = \frac{m^2}{1+\lambda A^2}. \tag{13}$$

Even the single-particle analogue of (10), ie the case of zero space dimensions, has interesting properties (Mathews and Lakshmanan 1974a), whose quantized version may also be solved exactly analytically (Mathews and Lakshmanan 1974b). It is assumed that in (13) when $\lambda < 0$, $|A| \leqslant |\lambda|^{-1/2}$. For $|A| > |\lambda|^{-1/2}$ the solution is well behaved but 'tachyonic' in nature, where the dispersion relation is of the form $(\omega^2 - k^2) = -m^2/(|\lambda|A^2 - 1) < 0$. However the field energy and momentum densities pertaining to this tachyonic solution become singular at various space–time points and so we discard this solution as physically uninteresting.

From the governing equation (10) we may derive the following set of two independent conservation equations (for simplicity we consider the one space dimensional case, assuming $(\boldsymbol{k} = k, 0, 0)$):

$$\frac{\partial \mathscr{H}}{\partial t} + \frac{\partial \mathscr{P}}{\partial x} = 0 \tag{14}$$

and

$$\frac{\partial \mathscr{P}}{\partial t} + \frac{\partial \mathscr{S}}{\partial x} = 0. \tag{15}$$

Here the energy density of the field is

$$\mathscr{H} = \frac{1}{2}\left(\frac{\phi_t^2 + \phi_x^2 + m^2\phi^2}{1 + \lambda\phi^2}\right), \tag{16}$$

the momentum density is

$$\mathscr{P} = -\frac{\phi_t \phi_x}{1 + \lambda\phi^2} \tag{17}$$

and

$$\mathscr{S} = \frac{1}{2}\left(\frac{\phi_t^2 + \phi_x^2 - m^2\phi^2}{1 + \lambda\phi^2}\right). \tag{18}$$

One may easily see that (14) and (15) are the energy–momentum conservation equations

$$\partial^\mu T_{\mu\nu} = 0$$

restricted to one space–one time dimensions, where the canonical energy–momentum tensor is given by

$$T_{\mu\nu} = (\partial_\mu \phi)\frac{\partial \mathscr{L}}{\partial(\partial^\nu \phi)} - g_{\mu\nu}\mathscr{L}.$$

Then in the spirit of Whitham the quantities $\mathscr{H}$, $\mathscr{P}$ and $\mathscr{S}$ in the exact equations (14) and (15) are replaced by their mean values in terms of the solutions (12) with (13) over a single wavelength. Thus we have the field energy and momentum per unit length as

$$\begin{aligned} \bar{\mathscr{H}} &= \frac{1}{2}\frac{1}{2\pi/k}\int_0^{2\pi/k} \frac{A^2[(\omega^2 + k^2)\cos^2(kx - \omega t) + m^2 \sin^2(kx - \omega t)]}{1 + \lambda A^2 \sin^2(kx - \omega t)}\, \mathrm{d}x \\ &= \frac{1}{2}\left(\frac{2k^2}{\lambda}[(1 + \lambda A^2)^{1/2} - 1] + \frac{m^2 A^2}{1 + \lambda A^2}\right), \end{aligned} \tag{19}$$

$$\bar{\mathscr{P}} = \frac{k}{\lambda}\left(k^2 + \frac{m^2}{1 + \lambda A^2}\right)^{1/2}[(1 + \lambda A^2)^{1/2} - 1], \tag{20}$$

while the average of $\mathscr{S}$ is

$$\bar{\mathscr{S}} = \frac{1}{\lambda}\left(k^2[(1 + \lambda A^2)^{1/2} - 1] + \frac{m^2}{(1 + \lambda A^2)^{1/2}} - \frac{m^2(2 + \lambda A^2)}{2(1 + \lambda A^2)}\right). \tag{21}$$

The nature of the field energy $\mathscr{E} = \overline{\mathscr{H}}$ is shown in figure 1 for both the signs of λ. It may be noted that when $\lambda < 0$, the field energy becomes infinite at finite amplitude ($|A| = |\lambda|^{-1/2}$), just as in the case of the $\lambda\phi^4$ field (Mathews and Lakshmanan 1973) when $\lambda < 0$. But unlike the $\lambda\phi^4$ case, the 'tachyonic' solutions here are not physically interesting as mentioned earlier.

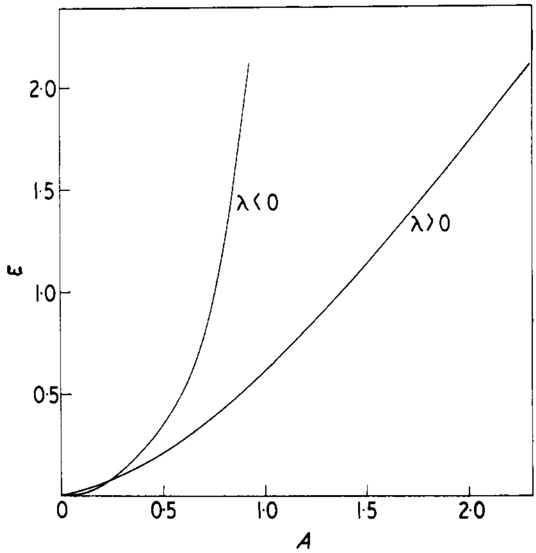

Figure 1. The field energy $\mathscr{E}$ as a function of the amplitude A (in units of $\lambda^{-1/2}$) for both $\lambda > 0$ and $\lambda < 0$.

Coming back to equations (14) and (15), the substitution of the mean values (19)–(21) gives us the following system of quasi-linear partial differential equations for k and A after some manipulations:

$$ak_t + bk_x + cA_t + dA_x = 0 \tag{22}$$

and

$$bk_t + ak_x + dA_t + eA_x = 0, \tag{23}$$

where

$$a = 2k,$$

$$b = \omega\left(1 + \frac{k^2}{\omega^2}\right), \tag{24}$$

$$c = \left(\frac{k^2(1+\lambda A^2)^{3/2} + m^2}{(1+\lambda A^2)^2}\right)\frac{\lambda A}{(1+\lambda A^2)^{1/2} - 1} \tag{25}$$

$$= \left(\frac{\omega^2}{(1+\lambda A^2)^{1/2} - 1} - \frac{m^2}{(1+\lambda A^2)^{3/2}}\right)\frac{\lambda A}{(1+\lambda A^2)^{1/2}} \tag{26}$$

$$d = \frac{k}{\omega}c, \tag{27}$$

and

$$e = \left(\frac{k^2}{(1+\lambda A^2)^{1/2} - 1} - \frac{m^2}{(1+\lambda A^2)^{3/2}}\right)\frac{\lambda A}{(1+\lambda A^2)^{1/2}}. \tag{28}$$

To verify the conservation of waves we proceed as below. Multiplying equation (22) by (k/ω) and subtracting from (23) we obtain

$$\left(a\frac{k}{\omega}-b\right)k_t+\left(b\frac{k}{\omega}-a\right)k_x+\left(d\frac{k}{\omega}-e\right)A_x=0 \tag{29}$$

which on simplification becomes

$$k_t+\left(\frac{k}{\omega}\right)k_x-\frac{\lambda m^2 A}{\omega(1+\lambda A^2)^2}A_x=0. \tag{30}$$

This is exactly equivalent to the equation

$$\frac{\partial k}{\partial t}+\frac{\partial\omega}{\partial x}=0,\qquad \omega=\left(k^2+\frac{m^2}{1+\lambda A^2}\right)^{1/2}. \tag{31}$$

One may think of k as the density of waves and ω the flux of the waves. Thus the conservation of waves automatically follows from the approximate conservation equations (22) and (23).

4. Propagation velocities

To investigate the nature of the equations (22) and (23) we make use of the method of characteristics and determine the nature of the characteristic roots. It is well known (Courant and Hilbert 1962, Jeffrey and Taniuti 1964) that for the system of first-order partial differential equations

$$\begin{aligned} a^{11}u_x^1+a^{12}u_x^2+b^{11}u_y^1+b^{12}u_y^2&=0,\\ a^{21}u_x^1+a^{22}u_x^2+b^{21}u_y^1+b^{22}u_y^2&=0 \end{aligned} \tag{32}$$

the characteristic curves C are given by the equation

$$\mathrm{d}x:\mathrm{d}y=\tau,\qquad\text{or}\qquad Q\left(x,y,\frac{\mathrm{d}x}{\mathrm{d}y}\right)=0, \tag{33}$$

where the characteristic roots τ are given by the determinantal equation

$$|Q|=\begin{vmatrix} a^{11}-\tau b^{11} & a^{12}-\tau b^{12}\\ a^{21}-\tau b^{21} & a^{22}-\tau b^{22}\end{vmatrix}=0. \tag{34}$$

If both the roots of (34) are real and distinct then the system is totally hyperbolic, while if both of them are complex the system is elliptic.

For our system (22) and (23) the characteristic equation becomes

$$\begin{vmatrix} b-\tau a & d-\tau c\\ a-\tau b & e-\tau d\end{vmatrix}=0. \tag{35}$$

On substitution of the values of a, b, c, d and e from equations (24)–(28), equation (35) becomes

$$[k^2(1+\lambda A^2)^{3/2}+m^2]\tau^2-2\omega k(1+\lambda A^2)^{3/2}\tau+\{k^2(1+\lambda A^2)^{3/2}+m^2[(1+\lambda A^2)^{1/2}-1]\}=0. \tag{36}$$

Thus the two characteristic roots of the system are given by

$$\tau = \frac{\omega k(1+\lambda A^2)^{3/2} \pm m^2[1-(1+\lambda A^2)^{1/2}]^{1/2}}{k^2(1+\lambda A^2)^{3/2}+m^2}. \tag{37}$$

We immediately see that when $\lambda > 0$ both the roots (37) are complex, making the system (22)–(23) elliptic. So the corresponding field modes are unstable in this case. This might be due to the fact that the total field energy is always less than that of the linear case (that is the $\lambda = 0$ limit). It is interesting to note that even in the case of the single-particle analogue of the lagrangian (1) (Mathews and Lakshmanan 1974a), the system executes aperiodic motions beyond a finite energy if $\lambda > 0$ and there are regions of instability for the periodic oscillations. When $\lambda < 0$, both in the single-particle and field (figure 2) cases, the total energy becomes infinite even at finite amplitude. This will have important repercussions in the quantum case too (Mathews and Lakshmanan 1974b). In the remaining we consider only the $\lambda < 0$ case.

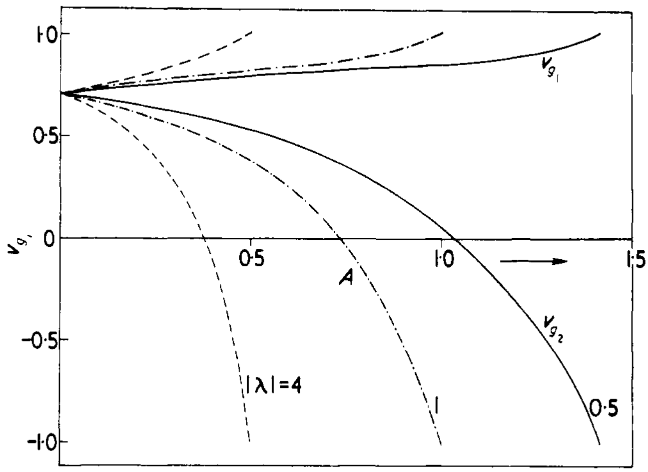

Figure 2. The group velocities v_{g_i} (for the $\lambda < 0$ case) as a function of the amplitude A for various values of λ (distinguished by different types of curves). In each of these cases, the maximum value of A is restricted in such a way that $|A| < |\lambda|^{-1/2}$.

When λ is negative (and $|A| < |\lambda|^{-1/2}$) the characteristic roots (37) are real and distinct and hence the system (22)–(23) is hyperbolic. Then the characteristic form (see for instance, Courant and Hilbert 1962, p 429) in this case is found to be

$$(k_t+\tau_i k_x) \pm \frac{\{1 \pm (k/\omega)[1-(1-|\lambda|A^2)^{1/2}]^{1/2}\}\omega}{[1-(1-|\lambda|A^2)^{1/2}]^{1/2}} \frac{|\lambda|A}{1-|\lambda|A^2}(A_t+\tau_i A_x) = 0 \tag{38}$$

or

$$\mathrm{d}k \pm \frac{\{1 \pm (k/\omega)[1-(1-|\lambda|A^2)^{1/2}]^{1/2}\}\omega}{[1-(1-|\lambda|A^2)^{1/2}]^{1/2}} \frac{|\lambda|A}{1-|\lambda|A^2}\mathrm{d}A = 0 \tag{39}$$

on the characteristic curves

$$C_\pm : \tau_i \equiv v_{g_i} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega k(1-|\lambda|A^2)^{3/2} \pm m^2[1-(1-|\lambda|A^2)^{1/2}]^{1/2}}{k^2(1-|\lambda|A^2)^{3/2}+m^2}. \tag{40}$$

Here $i = 1, 2$ correspond to the plus and minus signs in the numerator of the equations (38)–(40). The characteristic velocities v_{g_i} may be considered as the generalization of the group velocity of the linear case for our model (1). They are the velocities of propagation of changes in the simple harmonic wavetrain (12)†.

In the limit $|\lambda| \to 0$, we see that $v_{g_i} = \mathrm{d}x/\mathrm{d}t = k/\omega$ is a double characteristic along which k as well as the energy density are constants. In the nonlinear case (10) this splits up into two distinct characteristic velocities (if $\lambda < 0$) whose magnitudes are always less than 1 (as shown in figure 2).

To discuss the Riemann invariants, that is, functions which are constant along characteristics, it is more advantageous to talk in terms of the quantities $U = (k/\omega)$ and A. Then with the aid of the dispersion relation connecting ω, k and A, the characteristic form (39) may be re-expressed as

$$F(U)\,\mathrm{d}U \pm G(A)\,\mathrm{d}A = 0 \tag{41}$$

on the characteristic curves

$$C_\pm : \tau_i \equiv v_{g_i} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{U(1-|\lambda|A^2)^{1/2} \pm (1-U^2)[1-(1-|\lambda|A^2)^{1/2}]^{1/2}}{(1-|\lambda|A^2)^{1/2}U^2 + (1-U^2)} \tag{42}$$

$(i = 1, 2)$. Here

$$F(U) = (1-U^2)^{-1} \tag{43a}$$

and

$$G(A) = \frac{|\lambda|A}{[1-(1-|\lambda|A^2)^{1/2}]^{1/2}(1-|\lambda|A^2)}. \tag{43b}$$

Thus any wavetrain of the system (10) with $U = U_0$ and $A = A_0$ splits up after interaction into two simple waves (see for instance, Jeffrey and Taniuti 1964, p 69), one on the characteristic C_+ whose Riemann invariant from (41) is

$$\int_{U_0}^{U} F(U)\,\mathrm{d}U + \int_{A_0}^{A} G(A)\,\mathrm{d}A = \text{constant}, \tag{44}$$

along which the values of U and A are constants. The other wave is on the C_- characteristic whose Riemann invariant is the expression (44) with negative sign in front of the second term.

Finally, one may derive 'shock conditions' similar to problems in fluid and gas dynamics. However their significance is yet to be explored in the present case.

Acknowledgments

The author is indebted to Professor P M Mathews for his continuous guidance and constant encouragement. The financial assistance of the University Grants Commission, India in the form of a Junior Fellowship is gratefully acknowledged.

† These are also different from the energy flow velocity, ie the ratio of energy flux to energy density,

$$v_E = \frac{k[k^2+m^2(1-|\lambda|A^2)^{-1}]^{1/2}[(1-|\lambda|A^2)^{1/2}-1]}{k^2[(1-|\lambda|A^2)^{1/2}-1]-\frac{1}{2}|\lambda|m^2A^2(1-|\lambda|A^2)^{-1}}.$$

We note that in the $|\lambda| \to 0$ limit, $v_E \to k/\omega$.

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